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Convergence of the Complete Padé Tables of Trigonometric Functions*

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INTRODUCTION

The convergence properties of the Padé table of e^z are, in some sense, a model of regularity [3, pp. 244–248]. The proof of Padé's fundamental theorem relies so heavily on specific properties of the exponential function that, for more than 60 years, no extension of the result seems to have been noticed.

Two classes of functions, whose tables are as well behaved, have recently been studied: one of them by Arms and Edrei [1], the other by Edrei [2]. The present note may be considered as an application of both [1] and [2]. It shows that *real sine-polynomials and real cosine-polynomials whose zeros are all real* may be treated as completely as e^z.

Ratios of such trigonometric polynomials behave with similar regularity provided

(i) *all the zeros and poles of the ratio under consideration are simple*,

and

(ii) *zeros and poles are interlaced so that between any two zeros there lies exactly one pole and between any two poles exactly one zero.*

The simplest functions covered by our theorem are

$$(\cos z)^k, \qquad \left(\frac{\sin z}{z}\right)^k, \qquad \frac{\tan z}{z}, \qquad \frac{\tan^2 z}{z^2} \qquad (k \text{ an integer}),$$

and their reciprocals.

It is convenient to introduce at this point some notations and definitions. Their adoption will significantly shorten our statements and proofs.

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1. Notations, Definitions and Statement of Results

Let

$$\sum_{j=0}^{\infty} a_j z^j = A(z) \qquad (a_0 \neq 0), \tag{1.1}$$

be a power series.

Definition 1. Given an ordered pair (m, n) of nonnegative integers, we say that two polynomials $P_{mn}(z)$, $Q_{mn}(z)$ are Padé polynomials of the entry (m, n) if

(i) $Q_{mn}(z) \not\equiv 0, \qquad \text{degree } Q_{mn} \leqslant n;$

(ii) either $P_{mn}(z) \equiv 0$ or else degree $P_{mn} \leqslant m$;

(iii) $A(z)\, Q_{mn}(z) - P_{mn}(z) = z^{m+n+1} \mathscr{I}(z)$,

where $\mathscr{I}(z)$ is a series of nonnegative powers of z.

Padé polynomials of every entry always exist [3, pp. 235–236] and the rational function

$$R_{mn}(z) = P_{mn}(z)/Q_{mn}(z) \not\equiv 0$$

is uniquely determined by (m, n) and (1.1).

Placing $R_{mn}(z)$ in the nth row and mth column of an array, we obtain the Padé table of (1.1).

Set

$$a_{-j} = 0 \qquad (j = 1, 2, 3, \ldots), \tag{1.2}$$

and, with every pair (m, n), associate the polynomial

$$A_{mn}(z) = \begin{vmatrix} 1 & z & z^2 & \cdots & z^n \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix} \qquad (A_{m0}(z) \equiv 1), \tag{1.3}$$

and the Hankel determinant

$$A_m^{(n)} = A_{mn}(0). \tag{1.4}$$

If, for all $m \geqslant 0$, $n \geqslant 0$,

$$A_m^{(n)} \neq 0,$$

we say that *the table of* $A(z)$ *is normal.*

It is immediately verified that

$$A(z)\,A_{mn}(z) = \sum_{j=0}^{\infty} z^j \begin{vmatrix} a_j & a_{j-1} & a_{j-2} & \cdots & a_{j-n} \\ a_{m+1} & a_m & a_{m-2} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & \cdots & \cdots & a_m \end{vmatrix}, \tag{1.5}$$

which may be rewritten as

$$A(z)\,A_{mn}(z) - \mathcal{O}\!\mathcal{l}_{mn}(z) = (-1)^n A_{m+1}^{(n+1)} z^{m+n+1} + \cdots, \tag{1.6}$$

where $\mathcal{O}\!\mathcal{l}_{mn}(z)$ is the polynomial formed by those terms of the expansion (1.5) for which $0 \leqslant j \leqslant m$.

Definition 2. We say that the Padé polynomials of the entry (m, n) are essentially unique if

$$A_{mn}(z) \not\equiv 0. \tag{1.7}$$

The above definition and terminology are justified by the following remark [3, p. 236], which plays a fundamental role.

Remark 1. Let (1.7) hold and let P_{mn}, Q_{mn} be Padé polynomials of the entry (m, n). Then, there exists some constant $\xi = \xi_{mn} \neq 0$ such that

$$P_{mn}(z) = \xi \mathcal{O}\!\mathcal{l}_{mn}(z), \qquad Q_{mn}(z) = \xi A_{mn}(z).$$

Notational convention. Consider, besides $A(z)$, other power series,

$$B(z) = \sum_{j=0}^{\infty} b_j z^j, \qquad D(z), \ldots, T(z).$$

Expressions analogous to

$$A(z), \qquad A_m^{(n)}, \qquad A_{mn}(z), \qquad \mathcal{O}\!\mathcal{l}_{mn}(z),$$

obtained by replacing the $a_s{}'$ by $b_s{}', \ldots, t_s{}'$ will be denoted, respectively, by

$$\begin{array}{llll} B(z), & B_m^{(n)}, & B_{mn}(z), & \mathscr{B}_{mn}(z), \\ \cdots & \cdots & \cdots & \cdots \\ T(z), & T_m^{(n)}, & T_{mn}(z), & \mathscr{T}_{mn}(z). \end{array}$$

From this point on we take the above convention for granted and use it systematically without reminding the reader of the meaning of the symbols.

If the Padé polynomials are essentially unique, we use the following

NORMALIZATION. *Select $\alpha = \alpha_{mn} \neq 0$ so that 1 is the coefficient of the least power of z actually present in*

$$\tilde{A}_{mn}(z) = \alpha A_{mn}(z).$$

The polynomials $\tilde{A}_{mn}(z)$ and

$$\tilde{\mathcal{A}}_{mn}(z) = \alpha \mathcal{A}_{mn}(z)$$

are said to be the normalized Padé polynomials of the entry (m, n).

If $A_{mn}(z) \equiv 0$, the normalization process requires closer study [3, p. 237]. We need not discuss this singular case, which does not present itself here.

We denote, by

$$C(z) = u_0 + u_1 \cos z + u_2 \cos 2z + \cdots + u_N \cos Nz, \tag{1.8}$$

a real cosine-polynomial and require that all its zeros be real. [Multiple zeros are permissible.]

Similarly,

$$S(z) = w_1 \sin z + w_2 \sin 2z + \cdots + w_N \sin Nz \tag{1.9}$$

is a real sine-polynomial. We assume that all its zeros are real and do not exclude the possibility of multiple zeros.

The behavior of $C(z)$ and $S(z)$ at the origin is of some importance. In order to take account of it we "normalize" our trigonometrical sums and always write

$$A(z) = \frac{C(z)}{z^{2\mu}} = \sum_{j=0}^{\infty} a_j z^j \qquad (a_0 = 1,\ \mu \geqslant 0), \tag{1.10}$$

$$B(z) = \frac{S(z)}{z^{2\mu+1}} = \sum_{j=0}^{\infty} b_j z^j \qquad (b_0 = 1,\ \mu \geqslant 0). \tag{1.11}$$

Whenever we consider quotients such as

$$M(z) = S(z)/C(z), \tag{1.12}$$

we introduce additional restrictions and proceed as follows.

Let α_{k-1}, β_k $(k = 1, 2, 3,..., N)$ be $2N$ real quantities such that

$$0 = \alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_{N-1} < \alpha_{N-1} < \beta_N < \pi. \tag{1.13}$$

$$M(z) = \left[\kappa_1 \sin z \prod_{k=1}^{N-1} (\cos z - \cos \alpha_k)\right] \Big/ \left[\kappa_0 \prod_{k=1}^{N} (\cos z - \cos \beta_k)\right], \tag{1.14}$$

with

$$\kappa_1 = \prod_{k=1}^{N-1} (1 - \cos \alpha_k)^{-1}, \qquad \kappa_0 = \prod_{k=1}^{N} (1 - \cos \beta_k)^{-1}. \tag{1.15}$$

We put

$$T(z) = \frac{M(z)}{z} = \sum_{j=0}^{\infty} t_j z^j \qquad (t_0 = 1), \tag{1.16}$$

and also

$$V(z) = \left(\frac{\tan z}{z}\right)^2 = \sum_{j=0}^{\infty} v_j z^j \qquad (v_0 = 1). \tag{1.17}$$

We prove

THEOREM 1. *Let*

$$A(z), \qquad B(z), \qquad V(z), \qquad T(z) \tag{1.18}$$

be the functions defined above.

I. *Then all the Padé polynomials of the tables of* $A(z)$, $B(z)$, $V(z)$, *and* $T(z)$ *are essentially unique.*

II. *Let*

$$\{m_\lambda\}_{\lambda=1}^{\infty}, \qquad \{n_\lambda\}_{\lambda=1}^{\infty} \tag{1.19}$$

be two sequences of positive integers such that, as $\lambda \to \infty$,

$$m_\lambda \to \infty, \qquad n_\lambda \to \infty. \tag{1.20}$$

If λ *is restricted to those values for which* $m_\lambda n_\lambda$ *is even, we have, for the normalized Padé polynomials,*

$$\begin{gathered} \tilde{A}_{m_\lambda n_\lambda}(z) \to 1, \qquad \tilde{B}_{m_\lambda n_\lambda}(z) \to 1, \qquad \tilde{V}_{m_\lambda n_\lambda}(z) \to \cos^2 z, \\ \tilde{T}_{m_\lambda n_\lambda}(z) \to \kappa_0 \prod_{k=1}^{N} (\cos z - \cos \beta_k), \end{gathered} \tag{1.21}$$

and

$$\begin{gathered} \tilde{\mathscr{A}}_{m_\lambda n_\lambda}(z) \to A(z), \qquad \tilde{\mathscr{B}}_{m_\lambda n_\lambda}(z) \to B(z), \qquad \tilde{\mathscr{V}}_{m_\lambda n_\lambda}(z) \to \left(\frac{\sin z}{z}\right)^2, \\ \tilde{\mathscr{T}}_{m_\lambda n_\lambda}(z) \to \kappa_1 \frac{\sin z}{z} \prod_{k=1}^{N-1} (\cos z - \cos \alpha_k), \end{gathered} \tag{1.22}$$

uniformly in any bounded region of the complex plane.

If λ is restricted to those values for which $m_\lambda n_\lambda$ is odd, (1.21) *and* (1.22) *are to be replaced by similar relations in which the values of all the limits are multiplied by z.*

In the proof of the above result the behavior of the tables of $A(z)$, $B(z)$, and $V(z)$ will be deduced from the convergence theorem of Arms and Edrei [1, p. 4] concerning tables generated by sequences which are *totally positive* in the sense of Schoenberg [4, pp. 216–219]. The treatment of $T(z)$ depends on parts of the convergence theorem of [2]. An immediate application of the same convergence theorem leads to the following result which we state without proof.

THEOREM 2. *Let $\psi(z)$ be a real meromorphic function of finite order. Assume that*

(i) *$\psi(z)$ is periodic, with a real period;*

(ii) *the zeros and poles of $\psi(z)$ are all real, simple, and interlaced;*

(iii) $\psi(0) = 0$.

Consider the Padé table of the expansion

$$\frac{\psi(z)}{z} = U(z) = \sum_{j=0}^{\infty} u_j z^j,$$

and let $\{m(\lambda)\}_{\lambda=1}^{\infty}$, $\{n(\lambda)\}_{\lambda=1}^{\infty}$ satisfy the conditions (1.20) *as well as*

$$m(\lambda) + n(\lambda) = \textit{odd integer} \qquad (\lambda = 1, 2, 3, \ldots).$$

Then

$$U_m^{(n)} \neq 0 \qquad (m = m(\lambda),\ n = n(\lambda),\ \lambda = 1, 2, 3, \ldots)$$

and

$$\tilde{\mathscr{U}}_{mn}(z)/\tilde{U}_{mn}(z) \to U(z) \qquad (\lambda \to \infty)$$

uniformly on any compact set which omits the poles of $U(z)$.

Our conditions on $\psi(z)$ may be stated in the following, equivalent form. Normalize the period of $\psi(z)$ so that it is π and let the zeros α and the poles β, in $[0, \pi)$, be arranged so that

$$0 = \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \cdots < \alpha_N < \beta_N < \pi \qquad (N \geqslant 1).$$

Then the class of functions $\psi(z)$ coincides exactly with the class of functions of the form

$$K \prod_{j=1}^{N} \frac{\sin(z - \alpha_j)}{\sin(z - \beta_j)} \qquad (0 \neq K = \text{real constant}).$$

Our proofs require the following elementary remark, which may have some independent usefulness.

LEMMA 1. *Consider simultaneously, the expansions of the two functions*

$$D(z) \qquad (D(0) \neq 0), \qquad F(z) = D(z^k),$$

where $k \geqslant 1$ is an integer.

I. *Then*

$$F_{kp}^{(kq)} = \{D_p^{(q)}\}^k \tag{1.23}$$

for all pairs (p, q) of nonnegative integers.

II. *Assume that, for all $m \geqslant 0$ and all $n \geqslant 0$,*

$$D_m^{(n)} \neq 0, \tag{1.24}$$

and let $R_{mn}(z)$ denote the approximant of the entry (m, n) of the Padé table of $F(z)$. Then

$$R_{mn}(z) = \mathscr{D}_{pq}(z^k)/D_{pq}(z^k), \tag{1.25}$$

for all (m, n) of the $k \times k$-block defined by

$$kp \leqslant m \leqslant kp + k - 1, \qquad kq \leqslant n \leqslant kq + k - 1. \tag{1.26}$$

The approximant (1.25) *is never repeated in any other block.*

III. *If $k = 2$, and* (1.24) *holds, the Padé polynomials of all the entries of the table of $F(z) = D(z^2)$ are essentially unique and*

$$\begin{aligned} \tilde{F}_{2p,2q}(z) &\equiv \tilde{F}_{2p+1,2q}(z) \equiv \tilde{F}_{2p,2q+1}(z) \equiv \tilde{D}_{pq}(z^2), \\ \tilde{F}_{2p+1,2q+1}(z) &\equiv z\tilde{D}_{pq}(z^2). \end{aligned} \tag{1.27}$$

The analogous relations also hold for the Padé numerators so that (17) *remain true with F and D replaced by $\mathscr{F}$ and $\mathscr{D}$, respectively.*

For $k \geqslant 3$, Assertion III becomes more complicated. For instance, the Padé polynomials of the entries $(3p + 1, 3q + 1)$ of the table of $F(z) = D(z^3)$ are not essentially unique.

Exchanging the roles of the Padé numerators and denominators, we deduce from Theorem 1 the convergence properties of the tables of the reciprocals

$$1/A(z), \qquad 1/B(z), \qquad (z \cot z)^2, \qquad 1/T(z). \tag{1.28}$$

The replacement of z by iz would enable us to restate our results in terms of hyperbolic functions.

Our proof also yields

COROLLARY 1. *The Padé tables of the functions in* (1.18) *and* (1.28) *have the* 2×2*-block property described in Lemma* 1.

It is clear that Theorem 1 and its corollary solve completely the convergence problem of the Padé tables of the functions under consideration.

2. Proof of Assertion I of Lemma 1

We apply induction over k. Assume that (1.23) holds for some $k \geqslant 1$ and let

$$G(z) = D(z^{k+1}).$$

The first row of the determinant $G_{(k+1)p}^{((k+1)q)}$ is

$$d_p 00 \cdots 0 d_{p-1} 00 \cdots 0 d_{p-2} 00 \cdots 0 d_{p-q+1} 00 \cdots 0,$$

where each d is followed by k zeros. Consider the columns of this determinant headed by one of the q quantities,

$$d_p d_{p-1} \cdots d_{p-q+1}\,.$$

They form a matrix with $(k+1)q$ rows; only q of these rows contain d's; all others are formed exclusively by zeros. Hence, by Laplace's expansion theorem,

$$G_{(k+1)p}^{(k+1)q} = D_p^{(q)} F_{kp}^{(kq)}. \tag{2.1}$$

Since (1.23) is trivial for $k = 1$, (2.1) and an obvious induction show that (1.23) holds for all $k \geqslant 1$.

3. Proof of Assertion II of Lemma 1

Let m and n satisfy (1.26). Put

$$m = kp + \mu, \qquad n = kq + \nu, \qquad \tau = \min(\mu, \nu), \tag{3.1}$$

$$0 \leqslant \mu \leqslant k-1, \qquad 0 \leqslant \nu \leqslant k-1, \tag{3.2}$$

$$D_{pq}(z)\, D(z) - \mathscr{D}_{pq}(z) = (-1)^q\, D_{p+1}^{(q+1)}\, z^{p+q+1} + \cdots. \tag{3.3}$$

In (3.3), replace z by z^k and multiply the resulting relation by z^τ. This yields

$$z^\tau D_{pq}(z^k)\, F(z) - z^\tau \mathscr{D}_{pq}(z^k) = (-1)^q\, D_{p+1}^{(q+1)}\, z^{k(p+q)+k+\tau} + \cdots, \tag{3.4}$$

where, by (3.1) and (3.2),

$$m + n + 1 = k(p + q) + \mu + \nu + 1 \leqslant k(p + q) + k + \tau. \quad (3.5)$$

In view of (1.24),

$$z^{\tau} D_{pq}(z^k) \not\equiv 0,$$

and hence (3.5), the obvious relations

$$\text{degree}\{z^{\tau} D_{pq}(z^k)\} \leqslant kq + \tau \leqslant kq + \nu \leqslant n,$$
$$\text{degree}\{z^{\tau} \mathscr{D}_{pq}(z^k)\} \leqslant kp + \tau \leqslant kp + \mu \leqslant m,$$

and the uniqueness of the Padé table yield (1.25).

Assume now that two different blocks defined by (1.26) contain the same approximant. This means that there exist two distinct pairs of nonnegative integers, say (p, q) and (j, l), such that

$$\mathscr{D}_{pq}(\zeta)/D_{pq}(\zeta) \equiv \mathscr{D}_{jl}(\zeta)/D_{jl}(\zeta) \qquad (\zeta = z^k). \quad (3.6)$$

This is impossible because, by (1.24), the table of $D(z)$ is normal and (3.6) violates this normality [3, p. 243]. The proof of Assertion II of Lemma 1 is now complete.

4. Proof of Assertion III of Lemma 1

If

$$A_{mn}(z) \equiv 0, \quad (4.1)$$

the matrix obtained by deleting the first row of the determinant in (1.3) has rank $<n$.

Hence (4.1) implies

$$A_m^{(n)} = 0, \qquad A_{m+1}^{(n)} = 0,$$

and, in view of (1.5), also

$$A_m^{(n+1)} = 0, \qquad A_{m+1}^{(n+1)} = 0.$$

We thus see that (1.23) and (1.24) imply

$$F_{2p,2q}(z) \not\equiv 0, \qquad \text{because} \quad F_{2p}^{(2q)} = \{D_p^{(q)}\}^2 \neq 0,$$
$$F_{2p+1,2q}(z) \not\equiv 0, \qquad \text{because} \quad F_{2p+2}^{(2q)} = \{D_{p+1}^{(q)}\}^2 \neq 0,$$
$$F_{2p,2q+1}(z) \not\equiv 0, \qquad \text{because} \quad F_{2p}^{(2q+2)} = \{D_p^{(q+1)}\}^2 \neq 0,$$
$$F_{2p+1,2q+1}(z) \not\equiv 0, \qquad \text{because} \quad F_{2p+2}^{(2q+2)} = \{D_{p+1}^{(q+1)}\}^2 \neq 0.$$

We have thus proved that all the Padé polynomials of the table of $F(z) = D(z^2)$ are essentially unique. This uniqueness and the normalization which we have adopted readily yield (1.27) and complete the proof of Lemma 1.

5. The Padé Tables of $A(z)$ and $B(z)$

The most general real cosine-polynomial having all zeros real is of the form

$$C(z) = \kappa(\cos z - 1)^{\mu} \prod_{k=1}^{K} (\cos z - \cos \alpha_k)^{\mu_k}, \tag{5.1}$$

(i) where the α's are real and satisfy

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_K \leqslant \pi,$$

(ii) κ is a real constant;

(iii) μ_k $(k = 1, 2,..., K)$ are positive integers; $\mu \geqslant 0$ is an integer which is 0 if $C(0) \neq 0$.

The most general real sine-polynomial having all its zeros real is of the form

$$S(z) = C(z) \sin z, \tag{5.2}$$

where $C(z)$ is given by (5.1).

We set

$$A(z) = C(z)/z^{2\mu}, \qquad B(z) = S(z)/z^{2\mu+1}, \tag{5.3}$$

and select κ so as to satisfy the normalization

$$A(0) = 1 \qquad \text{or} \qquad B(0) = 1. \tag{5.4}$$

Consider the well-known product expansions

$$\sin z = z \prod_{l=1}^{\infty} \left(1 - \frac{z^2}{(l\pi)^2}\right), \tag{5.5}$$

$$\cos z - 1 = -2 \sin^2 \frac{z}{2} = -\frac{z^2}{2} \prod_{l=1}^{\infty} \left(1 - \frac{z^2}{(2l\pi)^2}\right)^2, \tag{5.6}$$

to which we add the slightly more general one,

$$\cos z - \cos \alpha = (1 - \cos \alpha) \prod_{l=-\infty}^{+\infty} \left(1 - \frac{z^2}{(2l\pi + \alpha)^2}\right) \quad (0 < \alpha \leqslant \pi). \tag{5.7}$$

We now introduce the new variable

$$\zeta = z^2 \tag{5.8}$$

and examine the auxiliary functions

$$D(\zeta) = \prod_{l=1}^{\infty} \left(1 - \frac{\zeta}{(2l\pi)^2}\right)^{-2\mu} \prod_{k=1}^{K} \prod_{l=-\infty}^{+\infty} \left(1 - \frac{\zeta}{(2l\pi + \alpha_k)^2}\right)^{-\mu_k} = \sum_{j=0}^{\infty} d_j \zeta^j, \tag{5.9}$$

and

$$G(\zeta) = D(\zeta) \prod_{l=1}^{\infty} \left(1 - \frac{\zeta}{(l\pi)^2}\right)^{-1} = \sum_{j=0}^{\infty} g_j \zeta^j. \tag{5.10}$$

It is immediately deduced from (5.1)–(5.10) that

$$A(z)\, D(z^2) \equiv 1, \qquad B(z)\, G(z^2) \equiv 1. \tag{5.11}$$

The sequences $\{d_j\}_{j=0}^{\infty}$ and $\{g_j\}_{j=0}^{\infty}$ are totally positive in the sense of Schoenberg [4, p. 219] and by the convergence theorem of Arms and Edrei [1, p. 4], we have

(i) $D_m^{(n)} > 0, \qquad G_m^{(n)} > 0,$

for all $m \geqslant 0$, $n \geqslant 0$;

(ii) if $\{p_\sigma\}_{\sigma=1}^{\infty}$ and $\{q_\sigma\}_{\sigma=1}^{\infty}$ are two sequences of positive integers such that, as $\sigma \to \infty$,

$$p_\sigma \to \infty, \qquad q_\sigma \to \infty, \tag{5.12}$$

then

$$\tilde{D}_{p_\sigma, q_\sigma}(\zeta) \to 1/D(\zeta), \qquad \tilde{\mathscr{D}}_{p_\sigma, q_\sigma}(\zeta) \to 1,$$
$$\tilde{G}_{p_\sigma, q_\sigma}(\zeta) \to 1/G(\zeta), \qquad \tilde{\mathscr{G}}_{p_\sigma, q_\sigma}(\zeta) \to 1,$$

uniformly in any bounded region of the ζ-plane.

Using Assertion III of Lemma 1 and (5.11), we obtain immediately the assertions of Theorem 1 concerning $A(z)$ and $B(z)$.

6. The Padé Table of $(\tan z/z)^2$

Consider two special cases of Theorem 1 which have been completely treated in the preceding section:

$$A(z) = \cos^2 z,$$

and

$$A^*(z) = [(1 - \cos z)/z^2](1 + \cos z) = [(\sin z)/z]^2.$$

We set

$$X(\zeta) = \prod_{l=1}^{\infty} \left(1 - \frac{4\zeta}{(2l-1)^2\pi^2}\right)^{-2} = \sum_{j=0}^{\infty} x_j \zeta^j, \tag{6.1}$$

$$Y(\zeta) = \prod_{l=1}^{\infty} \left(1 - \frac{\zeta}{l^2\pi^2}\right)^{-2} = \sum_{j=0}^{\infty} y_j \zeta^j, \tag{6.2}$$

so that

$$X(z^2)\cos^2 z \equiv 1, \qquad Y(z^2)[(\sin^2 z)/z^2] \equiv 1. \tag{6.3}$$

Introduce two auxiliary functions $H(\zeta)$ and $L(\zeta)$:

$$\frac{X(\zeta)-1}{\zeta} = H(\zeta) = \sum_{j=0}^{\infty} h_j \zeta^j = \sum_{j=0}^{\infty} x_{j+1}\zeta^j, \tag{6.4}$$

$$Y(\zeta) - \zeta = L(\zeta) = \sum_{j=0}^{\infty} l_j \zeta^j = 1 + (y_1 - 1)\zeta + \sum_{j=2}^{\infty} y_j \zeta^j. \tag{6.5}$$

In view of (6.3)

$$H(z^2) = (\tan^2 z)/z^2, \qquad L(z^2) = z^2/\tan^2 z, \tag{6.6}$$

and hence

$$H(\zeta)\,L(\zeta) \equiv 1 \qquad (H(0) = L(0) = 1). \tag{6.7}$$

Since $\{x_j\}_j$ and $\{y_j\}_j$ are totally positive, we have

$$H_m^{(n)} = X_{m+1}^{(n)} > 0 \qquad (m - n + 1 \geqslant 0), \tag{6.8}$$

$$L_n^{(m)} = Y_n^{(m)} > 0 \qquad (n - m + 1 \geqslant 2). \tag{6.9}$$

In view of (6.7) we have

$$H_m^{(n)} = (-1)^{mn}\, L_n^{(m)}. \tag{6.10}$$

[This is, in different notation, the relation (1.7) of [1, p. 8]. There is a misprint in (1.7); the correct relation is $a_0^{-n}A_m^{(n)} = b_0^{-m}B_n^{(m)}$.]

From (6.9) and (6.10) we deduce

$$H_m^{(n)} = (-1)^{mn}\, Y_n^{(m)} \neq 0 \qquad (-1 \geqslant m - n), \tag{6.11}$$

which, compared with (6.8), shows that the table of $H(\zeta)$ is normal. Hence, by Lemma 1, the table of

$$V(z) = H(z^2),$$

has the 2×2-block property and all its Padé polynomials are essentially unique.

To determine the Padé polynomials of the table of $H(\zeta)$ we first rewrite

$$X(\zeta)X_{m+1,n}(\zeta) - \mathscr{X}_{m+1,n}(\zeta) = (-1)^n X_{m+2}^{(n+1)} \zeta^{m+n+2} + \cdots$$

in the form

$$H(\zeta)\, X_{m+1,n}(\zeta) - [(\mathscr{X}_{m+1,n}(\zeta) - X_{m+1,n}(\zeta))/\zeta] = \chi\zeta^{m+n+1} + \cdots$$

and observe that, if

$$m + 1 - n \geqslant 0, \tag{6.12}$$

we have

$$X_{m+1,n}(\zeta) \equiv H_{mn}(\zeta)$$

and consequently

$$\tilde{H}_{mn}(\zeta) = \tilde{X}_{m+1,n}(\zeta); \qquad \tilde{\mathscr{H}}_{mn}(\zeta) = (\tilde{\mathscr{X}}_{m+1,n}(\zeta) - \tilde{X}_{m+1,n}(\zeta))/\zeta. \tag{6.13}$$

We have thus determined those normalized Padé polynomials of the table of $H(\zeta)$ whose entries are characterized by (6.12).

Similarly, for

$$n \geqslant m + 1, \tag{6.14}$$

$$\begin{aligned}(Y(\zeta) - \zeta)\, Y_{nm}(\zeta) &- (\mathscr{Y}_{nm}(\zeta) - \zeta Y_{nm}(\zeta)) \\ &= \frac{1}{H(\zeta)}\, Y_{nm}(\zeta) - (\mathscr{Y}_{nm}(\zeta) - \zeta Y_{nm}(\zeta)) = \chi' \zeta^{m+n+1} + \cdots\end{aligned}$$

and, since the Padé polynomials of the table of $H(\zeta)$ are essentially unique, there exists some constant $\xi \neq 0$ such that

$$\mathscr{Y}_{nm}(\zeta) - \zeta Y_{nm}(\zeta) = \xi \tilde{H}_{mn}(\zeta), \qquad Y_{nm}(\zeta) = \xi \tilde{\mathscr{H}}_{mn}(\zeta).$$

Hence

$$\tilde{H}_{mn}(\zeta) = \tilde{\mathscr{Y}}_{nm}(\zeta) - \zeta \tilde{Y}_{nm}(\zeta), \qquad \tilde{\mathscr{H}}_{mn}(\zeta) = \tilde{Y}_{nm}(\zeta), \tag{6.15}$$

provided (6.14) holds. Comparing (6.12) and (6.14), we see that all the Padé polynomials of all the entries of the table of $H(\zeta)$ are known.

Assume now that $\{p_\sigma\}_\sigma$, $\{q_\sigma\}_\sigma$ are sequences of positive integers satisfying the conditions (5.12). Then, by the convergence theorem of Arms and Edrei, we have, in view of (6.15) and (6.3),

$$\left.\begin{aligned}\tilde{H}_{p_\sigma q_\sigma}(z^2) &\to 1 - \frac{z^2}{Y(z^2)} = \cos^2 z \\ \tilde{\mathscr{H}}_{p_\sigma q_\sigma}(z^2) &\to \frac{1}{Y(z^2)} = \frac{\sin^2 z}{z^2}\end{aligned}\right\} \qquad (\sigma \to \infty,\ q_\sigma \geqslant p_\sigma + 1).$$

Similarly, using (6.13) instead of (6.15), we find

$$\left.\begin{aligned} \tilde{H}_{p_\sigma q_\sigma}(z^2) &\to \frac{1}{X(z^2)} = \cos^2 z \\ \tilde{\mathscr{H}}_{p_\sigma q_\sigma}(z^2) &\to \frac{1-(1/X(z^2))}{z^2} = \frac{\sin^2 z}{z^2} \end{aligned}\right\} \qquad (\sigma \to \infty,\ p_\sigma + 1 \geqslant q_\sigma).$$

From Lemma 1, we immediately deduce the convergence properties of the normalized Padé polynomials of $(\tan z/z)^2$.

7. Quotients of Trigonometric Polynomials Expressed as Sums of Simple Fractions

The quotient of trigonometric polynomials $M(z)$, defined by (1.13) and (1.14), has period 2π and, in the strip

$$-\pi \leqslant x < \pi \qquad (x = \operatorname{Re} z),$$

it has exactly $2N$ simple poles:

$$\pm\beta_1,\ \pm\beta_2,\ldots,\ \pm\beta_N.$$

At $-\beta_j$ and β_j it has the same residue r_j given by

$$r_j = -\left[\kappa_1 \prod_{k=1}^{N-1} (\cos\beta_j - \cos\alpha_k)\right] \Big/ \left[\kappa_0 \prod_{\substack{k=1 \\ k\neq j}}^{N} (\cos\beta_j - \cos\beta_k)\right] = -\rho_j < 0. \tag{7.1}$$

The fact that $r_j < 0$ follows immediately from (7.1) and the interlacing of zeros and poles expressed by (1.13).

The elementary expansion

$$\begin{aligned} \frac{1}{2}\left\{\cot\left(\frac{z-\beta_k}{2}\right) + \cot\left(\frac{z+\beta_k}{2}\right)\right\} &= \sum_{l=-\infty}^{+\infty} \left\{\frac{1}{z-\beta_k-2\pi l} + \frac{1}{z+\beta_k+2\pi l}\right\} \\ &= 2z \sum_{l=-\infty}^{+\infty} \frac{1}{z^2 - (\beta_k + 2\pi l)^2} = \phi_k(z) \end{aligned}$$

shows that

$$M(z) - \sum_{k=1}^{N} r_k \phi_k(z) = \Omega(z)$$

is entire, periodic, and bounded outside disks of positive fixed radius with centers at $\pm\beta_k + 2l\pi$ ($k = 1, 2,\ldots, N$; $l = 0, \pm 1, \pm 2, \pm 3,\ldots$).

Hence $\Omega(z)$ reduces to a constant which is necessarily zero because $\Omega(z)$ is an odd function.

Returning to $T(z)$ defined by (1.16) and setting again

$$z^2 = \zeta,$$

we find

$$T(z) = \frac{M(z)}{z} = 2 \sum_{k=1}^{N} r_k \sum_{l=-\infty}^{+\infty} \frac{1}{\zeta - (\beta_k + 2\pi l)^2}$$

$$= 2 \sum_{l=-\infty}^{+\infty} \sum_{k=1}^{N} \frac{\rho_k}{(\beta_k + 2\pi l)^2 - \zeta} = Z(\zeta), \tag{7.2}$$

where, by (7.1), $\rho_k > 0$ $(k = 1, 2,..., N)$.

8. The Padé Tables of $Z(\zeta)$ and $T(z)$

The positive quantities

$$(\beta_k + 2\pi l)^2 \qquad (k = 1, 2,..., N;\ l = 0, \pm 1, \pm 2,...) \tag{8.1}$$

may be arranged and renumbered so that they form a single increasing sequence

$$\omega_1, \omega_2, \omega_3, \dots . \tag{8.2}$$

We modify correspondingly the notation of the quantities $2\rho_k$ and rewrite (7.2) in the form

$$T(z) = \frac{M(z)}{z} = \sum_{k=1}^{\infty} \frac{\psi_k}{\omega_k - \zeta} = Z(\zeta), \tag{8.3}$$

where

$$\psi_k > 0 \quad (k = 1, 2, 3,...), \qquad \sum_{k=1}^{\infty} \frac{\psi_k}{\omega_k} < +\infty, \qquad \sum_{k=1}^{\infty} \frac{1}{\omega_k} < +\infty. \tag{8.4}$$

The relations (8.3) and (8.4) enable us to apply the convergence theorem of [2] which asserts the following.

I. There exists a positive sequence $\{\gamma_k\}_{k=1}^{\infty}$ such that

$$\omega_1 < \gamma_1 < \omega_2 < \gamma_2 < \cdots < \omega_k < \gamma_k < \omega_{k+1} < \cdots$$

and such that

$$Z(\zeta) = \prod_{k=1}^{\infty} \left(1 - \frac{\zeta}{\gamma_k}\right) \Big/ \prod_{k=1}^{\infty} \left(1 - \frac{\zeta}{\omega_k}\right). \tag{8.5}$$

II. The Padé table of $Z(\zeta)$ is normal.

III. If $\{p_\sigma\}_{\sigma=1}^{\infty}$ and $\{q_\sigma\}_{\sigma=1}^{\infty}$ are two sequences of positive integers satisfying the conditions (5.12), we have for the normalized Padé polynomials of $Z(\zeta)$:

$$\tilde{Z}_{p_\sigma,q_\sigma}(\zeta) \to \prod_{k=1}^{\infty}\left(1-\frac{\zeta}{\omega_k}\right), \tag{8.6}$$

$$\tilde{\mathscr{Z}}_{p_\sigma,q_\sigma}(\zeta) \to \prod_{k=1}^{\infty}\left(1-\frac{\zeta}{\gamma_k}\right). \tag{8.7}$$

The convergence in (8.6) and (8.7) is uniform in any bounded region of the complex plane.

Since the set (8.1) coincides with the sequence $\{\omega_k\}_{k=1}^{\infty}$, we have by (1.14), (1.15), (1.16), and (5.7)

$$\kappa_0 \prod_{k=1}^{N} (\cos z - \cos \beta_k) = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{\omega_k}\right),$$

$$\kappa_1 \frac{\sin z}{z} \prod_{k=1}^{N-1} (\cos z - \cos \alpha_k) = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{\gamma_k}\right).$$

We now use Lemma 1 exactly as in the corresponding proofs at the end of Sections 5 and of 6. This yields the behavior of the Padé table of $T(z)$ and completes the proof of Theorem 1.

References

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